

Three classes of triangles

1 Определения и вводные задачи

2 Main problems

Solutions.

13A. a) **Answer.** H'_a .

Hint 1. For any triangle consider a transformation that is the composition of inversion with center A and radius $\sqrt{AB \cdot AC}$ and symmetry with respect to the bisector of angle A . It maps B to C , and the line BC to Ω . Hence it maps γ_a to ω_a . Thus line through A and touching points of ω_a and Ω is symmetrical to AA_a in AI . But in A -triangle altitude is symmetrical to AA_a in AI .

Hint 2. Touching point of ω_a and Ω is the center of homothety with a positive ratio that maps one of these circles to other. Consider the incircle, and use the result of problem 11A and theorem on three centers of homothety.

b) It is known that in any triangle line connecting tangency point ω_a and Ω with point A'' , bisects segment $B'C''$. Since arcs $B'A''$ and AC'' are equal, H'_aA is a symmedian of triangle $H'_aB'C''$. This is equivalent to the statement of the problem.

14A. Easily follows from the fact that the sum of radii is $BC \cos B + BC \cos C = BC$.

15A. Since $BX : CX = \cos B : \cos C = BH'_a : CH'_a$, we obtain that X lies in the bisector H'_aA'' of triangle H'_aBC . Now the statement follows from the fact: in any triangle the line through A'' and the tangency point of Ω and ω_a passes through I .

16A. Lines IH and H'_aA' are external bisectors of triangles HBC and H'_aBC that are symmetric in BC .

17A. Since HX и HY are internal and external bisector of the angle BHC , the quadruple B, C, X, Y is harmonic. We know that B and C — are centers of circles Γ_b, Γ_c , and X — is the center of their homothety with negative ratio. Hence Y is the center of their homothety with positive ratio.

18A. Line H'_aA_1 passes through O , hence it is symmetric to H'_aA with the respect of the bisector H'_aI of the angle BH'_aC (since H'_aA is the altitude of triangle BH'_aC). The statement of the problem follows from the fact that X also lies in this bisector (a tangent to γ through X passes through the point that is symmetric to A_1 in H'_aI). The proof for Y is analogous.

19A. In any triangle line HA_0 passes through the point A_2 of Ω such that AA_2 is a diameter. Hence the required statement to equality $(BCA_0X) = (BCA_2A')$ that can be obtained by easy calculation.

Let P be the second common point of $A'X$ and Ω . Then $\angle A''PA' = 90^\circ$, thus line $A''P$ passes through Y . Now $\angle A'PA_0 = \angle A'XA_0 = \angle H'_aA' = \angle A'PA_2$, hence A_0 lies in PA_2 .

- 20A. Consider triangles $OA'A_2$ and H'_aIH . We know that $A'O$ meets H'_aI at A'' . Further, it is clear that H'_aH and A_2O meet at A . Sidelines HI and A_2A' meet at the infinite point of $A''A$ since lines AA'' , HI и $A'A_2$ are perpendicular to AI . Now the statement follows from the Desargues theorem.
- 21A. a) Lines H'_aO and AA' pass through the center of ω_a because H'_a and A are homothety centers of ω_a with Ω and γ respectively.
- b) Let K be the common point of AO and H'_aI . Then K and T are the feet of the bisectors in isosceles triangle AOH'_a . From this we obtain that $AK = KT = TH'_a$, hence TK is the radius of ω_a . It is clear that line OI is the perpendicular bisector of TK , therefore the answer is 120° .
- 6B. In any triangle B_0I intersects BH at point Z such that $BZ = r$. In particular, for B -triangle we have $BH = 2R \cos B = r_b \cos B = r$.
- 7B. I lies in lines C_bA_1 and A_bC_1 , hence $IA_bI_bC_b$ is a rhombus, therefore, the midpoint of A_bC_b coincides with the midpoint B' of II_c .
- 8B. Follows from the solution of 5B.
- 9B. From 5B it follows that the projections of Bb to AB and BC are C_c and A_a , respectively. Hence A_a, B_b, C_c, H_b belong to the circle with diameter BB_b .
- 9C. A_c, B_c, C_c are intersection points of the opposite sidelines and diagonals of the quadrilateral $ACBG_c$ inscribed to Ω . Hence A_c, B_c, C_c are the vertices of a triangle autopolar with respect to Ω .
- 10C. AB passes through the pole C_c of the line A_cB_c , therefore the pole of AB lies in A_cB_c .
- 11C. From $\cos C = \cos A + \cos B$ we obtain that the distance from O to AB equals the sum of distances from O to BC and CA . Points L_a, L_b also satisfy this condition. Now it suffices to note that this linear condition defines a line.
- 12C. It is known that the line L_aL_b is fixed for all triangles with fixed circles Ω and γ_c . For C -triangles this line is a diameter of Ω perpendicular to OI_c . The endpoints of this diameter and I_c are the vertices of an equilateral triangle whose center lies in OI_c and divides the segment OI_c with ratio $1 : 2$.
- 13C. Triangles ABC and H_aH_bC similar with ratio $\cos C$, hence H_aH_b intersects the bisector of the angle C with ratio $\cos C : (1 - \cos C)$. From the property of bisectors and equality $\cos C = \cos A + \cos B$ it is easy to prove that I divides the bisector with the same ratio.

- 14C. The bisector of the angle C is the bisector of the angle OCH_c . Hence it intersects the segment OH_c at a point V with $OV : VH_c = R : CH_c$. The distanced from H_c to BC and AC equal $CH_c \cos B$ and $CH_c \cos A$, respectively. From this we obtain that the condition $\cos C = \cos A + \cos B$ holds iff $V = I$.
- 15C. Since O lies in $L_a L_b$, one can reformulate the statement of the problem in the following way: IL_a , IL_b , IO , and the perpendicular from I to OC is a harmonic quadruple of lines. From two previous problems it follows that these lines intersect AB at A , B , H_c , and the common point of AB and $H_a H_b$.
- 16C, 17C. **Hint.** When C moves on Ω L moves on the hyperbola touching the sidelines of all triangles $A_c B_c C_c$.